Yamabe solitons and concurrent vector fields

Abstract

In this paper we consider Yamabe soliton and we will show that if the potential vector field of a Yamabe soliton be a concurrent vector field then the scalar curvature of the underlying manifold is a real constant. Also, we obtain bound on the soliton constant. Finally, we show that there is not any compact Yamabe soliton with concurrent field.

Keywords: Riemannian geometry, Concurrent vector fields, Yamabe soliton.

1 Introduction

It is shown that if the holonomy group of Riemannain manifold of dimension $n$ leaves a point invariant, then there exists a smooth vector field $X$ on $M$ which satisfies

$$\nabla_Y X = Y,$$

(1.1)

for any vector $Y$ tangent to $M$, where $\nabla$ stands for the Levi-Civita connection of the underlying manifold $M$. (For more details you can consult [5].) Such vector fields is called concurrent vector fields. Riemannian geometry as well as Finsler geometry of manifolds equipped with concurrent vector fields have been studied in many author experiments, so far. (For example, see [1], [3], [4] and [2].) Also, we will see that the concurrent vector fields are responsible for some interesting properties of manifolds.

In the other hand, the notion of Yamabe flow was introduced by R.S. Hamilton in 1988 in order to study Yamabe’s conjecture stating that any metric is conformally related to a metric with constant scalar curvature. Also, Yamabe solitons serve as self similar solutions of Yamabe flow. This notion is the subject of many researches in the last decade. A Riemannian manifold $(M^n, g)$ is said to be a Yamabe soliton if there are a smooth vector field $X$ and a constant $\lambda$ such that

$$\mathcal{L}_X g = 2(R + \lambda)g,$$

(1.2)

where $R$ is the scalar curvature and $\mathcal{L}$ stands for the Lie derivative operator. The soliton is shrinking, steady and expanding according as $\lambda < 0$, $= 0$ and $> 0$. If $X = \nabla f$ for some real valued smooth function $f$ on $M$, then it is called the gradient Yamabe soliton and $f$ is called the potential function. In this case the equation (1) can be rewritten as follows.

$$\nabla^2 f = (R + \lambda)g.$$ 

(1.3)
In the present paper, we consider the above geometric structures together. A Yamabe soliton \((M^n, g, X, \lambda)\) on a Riemannian manifold \((M^n, g)\) is said to have concurrent potential field if its potential field \(X\) is a concurrent vector field. Now, we prepare to present our main results.

**Theorem 1.1** If a Yamabe soliton \((M, g, X, \lambda)\) on a Riemannian manifold \((M^n, g)\) has concurrent potential field \(X\) then, the following conditions hold,

a) the scalar curvature \(R\) is a real constant,

b) if the scalar curvature is bounded below by some constant \(\alpha\), then \(1 - \alpha\) is the upper bound for the soliton constant.

The following corollary shows that there are expanding, steady or shrinking Yamabe soliton with concurrent potential fields.

**Corollary 1.2** For a Yamabe soliton \((M, g, X, \lambda)\) with concurrent potential vector field we have

a) if the scalar curvature \(R > 1\), the soliton is shrinking,

b) if the scalar curvature \(R = 1\), the soliton is steady.

Now, we show that there is no compact Yamabe soliton with concurrent potential fields.

**Theorem 1.3** There is no compact Yamabe soliton with concurrent potential vector field.

## 2 Proof of main results

In this section we prove our main results.

**Proof of Theorem 1.1:** Let \((M, g, X, \lambda)\) is a Yamabe soliton on a Riemannian manifold \(M\) of dimension \(n\). Also, assume that \(X\) is a concurrent vector field, then we have

\[ \nabla_Y X = Y, \quad \forall Y \in TM. \]  

(2.1)

According to the definition of Lie-derivative and the above condition we derive

\[ (\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 2g(Y, Z), \]  

(2.2)

for every \(X, Y\) tangent to \(M\). Combining the above formula with Yamabe soliton equation (1.2) gives

\[ (\lambda + R - 1)g = 0, \]  

(2.3)

which shows that \(R = 1 - \lambda\) is a real constant. So, if the scalar curvature is bounded below by some constant \(\alpha\), then the soliton constant will be bounded above by \(1 - \alpha\) and Corollary 1.2 is immediately obtained.

**Proof of Theorem 1.3:** Let \((M^n, g, X, \lambda)\) is a compact Yamabe soliton. Tracing (1.2) yields

\[ \text{div} X = (R + \lambda)n. \]
Since the integral of $\text{div} X$ vanishes, so we have

$$\int_M n(R + \lambda)\Omega_g = 0,$$

where $\Omega_g$ is the volume form related to the metric $g$. Hence $R + \lambda = 0$. But, as mentioned above, $R = 1 - \lambda$, so

$$1 - \lambda = -\lambda$$

which is a contradiction.

References


